Journal of Statistical Physics, Vol. 48, Nos. 3/4, 1987

Self-Synchronization of Populations of Nonlinear Oscillators in the Thermodynamic Limit

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Received November 25, 1986; revision received March 10, 1987

A population of identical nonlinear oscillators, subject to random forces and coupled via a mean-field interaction, is studied in the thermodynamic limit. The model presents a nonequilibrium phase transition from a stationary to a time-periodic probability density. Below the transition line, the population of oscillators is in a quiescent state with order parameter equal to zero. Above the transition line, there is a state of collective rhythmicity characterized by a time-periodic behavior of the order parameter and all moments of the probability distribution. The information entropy of the ensemble is a constant both below and above the critical line. Analytical and numerical analyses of the model are provided.

KEY WORDS: Nonequilibrium phase transitions; self-synchronization; nonlinear Fokker–Planck equation; Hopf bifurcation.

1. INTRODUCTION

The appearance of temporal order in macroscopic systems out of equilibrium is a much analyzed phenomenon in physics and other sciences. This temporal ordering is often a cooperative effect arising in large collections of interacting nonlinear subsystems in contact with a thermal bath. Among processes of temporal ordering, the self-synchronization of large populations of oscillators is particularly conspicuous.[1–6] The purpose of the present paper is to illustrate this process in a simple model, where detailed calculations are possible. We analyze an ensemble of infinitely many identical nonlinear oscillators interacting via a linear mean-field term and subject to thermal noise. The model presents a nonequilibrium phase transition from a stationary to a time-periodic probability density. Below

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0022-4715/87/0800-0571$05.00/0

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the transition line, the population of oscillators is in a quiescent state with order parameter equal to zero. Above the critical line, there is a state of collective rhythmicity characterized by a time-periodic behavior of the order parameter and all moments of the probability distribution.

The search for an adequate statistical description of collective rhythmicity has been encouraged by the abundance of examples in biology, chemistry, etc.[1-6] Early attempts by Yamaguchi *et al.*[1] and Kuramoto[2] emphasized the dynamical aspects of the problem: They considered a model of the type described above as basically a dynamical system perturbed by noise and by coupling among subsystems. Thus, these authors understated the statistical features of the models. A statistical treatment was provided by Desai and Zwanzig[3] for a model of interacting identical particles:

$$dx_j/dt = (1 - x_j^2)x_j + F^{1/2}w_j(t) - J\left[x_j - N^{-1}\sum_{1 \leqslant k \leqslant N} x_k\right], \qquad j = 1,..., N \tag{1.1}$$

Here the $w_j(t)$ are independent Gaussian white noises $[\langle w_j \rangle = 0, \langle w_j(t)\, w_k(t') \rangle = \delta_{jk}\, \delta(t - t')]$. With $J = 0$, Eq. (1.1) describes the relaxation of a particle in a double-well potential in contact with a thermal bath at temperature F. The last term in (1.1) can be viewed as an interaction between particles, which creates a tendency for their coordinates x_j to relax toward the center of mass of the ensemble. Equation (1.1) was introduced by Kometani and Shimizu[4] to illustrate the formal treatment later used in Ref. 1. Desai and Zwanzig derived the following nonlinear Fokker–Planck equation (FPE) for the one-particle probability density:

$$\partial_t\, p(t; x) = \tfrac{1}{2}F\, \partial_x^2\, p(t; x) - \partial_x\{[(1 - x^2)x + J(\langle x(t) \rangle - x)]\, p(t; x)\} \tag{1.2a}$$

$$\langle x(t) \rangle = \int x p(t; x)\, dx \tag{1.2b}$$

$$\int p(t; x)\, dx = 1 \tag{1.2c}$$

They used the molecular chaos assumption $[p_2(t; x_1, x_2) = p(t; x_1)\, p(t; x_2)]$ to close a hierarchy of equations for all the multiparticle probability densities. Desai and Zwanzig also gave another derivation which made clear that (1.2) is asymptotically valid in the limit $N \to \infty$.[3] Later, Dawson proved this.[5] An analysis of (1.2) showed the existence of a pitchfork bifurcation for its stationary solutions, which corresponds to an equilibrium phase transition. Furthermore, the approach to equilibrium in the thermodynamic limit can be studied from (1.2).[3,5,6]

Nonlinear Fokker–Planck equations similar to (1.2) have been derived for more general mean-field models.[7] For different models, Bonilla[7] found stable time-dependent solutions of these nonlinear FPE. An oscillatory probability density is a natural candidate for the state of collective rhythmicity that characterizes oscillator self-synchronization. To study self-synchronization of nonlinear oscillators, we need subsystems with more than one degree of freedom. One of the simplest such models is the following:

$$d\mathbf{x}_j(t)/dt = (\alpha - \mathbf{x}_j^2)\mathbf{x}_j + \mathbf{x}_j^\dagger + F^{1/2}\mathbf{w}_j(t) - J\left[\mathbf{x}_j(t) - N^{-1}\sum_{1\leqslant k\leqslant N}\mathbf{x}_k(t)\right] \tag{1.3a}$$

$$\mathbf{x}_j \equiv (x_j,\ y_j), \qquad \mathbf{x}_j^\dagger \equiv (-y_j,\ x_j), \qquad \mathbf{x}_j \cdot \mathbf{x}_j^\dagger = 0 \tag{1.3b}$$

Here $j = 1,..., N$. For $J = F = 0$, the stable solution of (1.3a) is $\mathbf{x}_j = \mathbf{0}$ if $\alpha < 0$ and $\mathbf{x}_j = \alpha^{1/2}[\cos(t+\beta_j), \sin(t+\beta_j)]$ if $\alpha > 0$ $\{\beta_j = \tan[y_j(0)/x_j(0)]\}$. Thus, for $\alpha > 0$, (1.3) is a collection of nonlinear oscillators subject to thermal noise fluctuations and coupled via a mean-field interaction. In the limit $N \to \infty$, the one-oscillator probability density $p(t, \mathbf{x})$ corresponding to (1.3) exhibits first- and second-order nonequilibrium phase transitions. Below the line corresponding to second-order phase transitions, the only stable density is stationary, with order parameter $\langle \mathbf{x} \rangle = \mathbf{0}$. This represents a quiescent state, with the oscillators moving out of phase. Above the transition line, the stable probability density is time-periodic, thereby representing a state of collective rhythmicity: the oscillators move in phase. Near the line corresponding to first-order phase transitions, stable stationary and oscillatory densities coexist, and hysteresis loops are therefore possible. Furthermore, the information entropy, defined as $-\int p(t, \mathbf{x}) \ln p(t, \mathbf{x})\, d\mathbf{x}$, is time-independent for the stable time-periodic density. Some of these results were announced in Ref. 7. We derive them in Section 2, together with several new results. They are numerically illustrated in Section 3. Section 4 consists of a summary of our findings. The Appendix is devoted to the derivation of several technical results needed in the text.

2. ANALYSIS OF THE MODEL

Asymptotically as $N \to \infty$, the one-oscillator probability density $p(t, \mathbf{x})$ corresponding to (1.3) obeys the following nonlinear Fokker–Planck equation[7]:

$$\partial_t p(t; \mathbf{x}) = \tfrac{1}{2}F\,\Delta p(t; \mathbf{x}) - \nabla \cdot \{[(\alpha - \mathbf{x}^2)\mathbf{x} + \mathbf{x}^\dagger + J(\langle \mathbf{x}(t) \rangle - \mathbf{x})]\, p(t; \mathbf{x})\} \tag{2.1a}$$

$$\langle \mathbf{x}(t) \rangle = \int \mathbf{x} p(t; \mathbf{x})\, d\mathbf{x} \tag{2.1b}$$

$$\int p(t, \mathbf{x})\, d\mathbf{x} = 1 \tag{2.1c}$$

$$p(0, \mathbf{x}) = \mu(\mathbf{x}) \tag{2.1d}$$

We have supposed a molecular chaos initial condition for the N-oscillator probability density[7]:

$$P_N(0; \mathbf{x}_1, ..., \mathbf{x}_N) = \prod_1^N \mu(\mathbf{x}_j)$$

Let us make the substitution $p(t, \mathbf{x}) = \exp[-\Phi(t, \mathbf{x})]$ in the nonlinear Fokker–Planck equation (2.1). The resulting nonlinear equation is

$$\begin{aligned} -\partial_t \Phi(t, \mathbf{x}) = &-[\tfrac{1}{2} F \Delta \Phi(t, \mathbf{x}) + \nabla \cdot \mathbf{g}(\mathbf{x})] + \tfrac{1}{2} F [\nabla \Phi(t, \mathbf{x})]^2 \\ &+ [\mathbf{g}(\mathbf{x}) + J \langle \mathbf{x}(t) \rangle] \cdot \nabla \Phi(t, \mathbf{x}) \end{aligned} \tag{2.2a}$$

$$\mathbf{g}(\mathbf{x}) = (\alpha - J - \mathbf{x}^2)\mathbf{x} + \mathbf{x}^\dagger \tag{2.2b}$$

The following stationary solution solves $\frac{1}{2} F \Delta \Phi(t, \mathbf{x}) + \nabla \cdot \mathbf{g}(\mathbf{x}) = 0$:

$$\begin{aligned} p_s(\mathbf{x}) &= Z^{-1} \exp[-\phi(\mathbf{x})] \\ \phi(\mathbf{x}) &= (J - \alpha + \tfrac{1}{2}\mathbf{x}^2)\mathbf{x}^2 / F \\ Z &= \int \exp[-\phi(\mathbf{x})]\, d\mathbf{x} \end{aligned} \tag{2.3}$$

The solution (2.3) also solves (2.2) with $\langle \mathbf{x} \rangle = \mathbf{0}$. It is a stationary state of type II, according to Jauslin's classification scheme[8] for the FPE. The change of variable $p(t, \mathbf{x}) = p_s(\mathbf{x}) + [p_s(\mathbf{x})]^{1/2} q(t, \mathbf{x})$ in (2.1) results in the following nonlinear equation for $q(t, \mathbf{x})$:

$$\partial_t q(t, \mathbf{x}) = \mathbf{L} q + J[\tfrac{1}{2} q(t, \mathbf{x}) \nabla \phi(\mathbf{x}) - \nabla q(t, \mathbf{x})] \cdot \langle \mathbf{y} [p_s(\mathbf{y})]^{1/2}, q(t, \mathbf{y}) \rangle \tag{2.4a}$$

$$\mathbf{L} q \equiv (L_H + L_A)\, q(t, \mathbf{x}) + J[p_s(\mathbf{x})]^{1/2} \nabla \phi(\mathbf{x}) \cdot \langle \mathbf{y} [p_s(\mathbf{y})]^{1/2}, q(t, \mathbf{y}) \rangle \tag{2.4b}$$

$$L_H q(t, \mathbf{x}) \equiv \tfrac{1}{2} F \Delta q(t, \mathbf{x}) + \tfrac{1}{4} F \{ \Delta \phi(\mathbf{x}) - \tfrac{1}{2} [\nabla \phi(\mathbf{x})]^2 \}\, q(t, \mathbf{x}) \tag{2.4c}$$

$$L_A q(t, \mathbf{x}) \equiv -\mathbf{x}^\dagger \cdot \nabla q(t, \mathbf{x}) \tag{2.4d}$$

$$\langle f(\mathbf{y}), g(\mathbf{y}) \rangle \equiv \int f(\mathbf{y})\, g(\mathbf{y})\, d\mathbf{y} \tag{2.4e}$$

The operators L_H and L_A are Hermitian and anti-Hermitian, respectively. They also commute. Let us consider for a moment the linear FPE that results from setting $\langle \mathbf{x}(t) \rangle \equiv \mathbf{0}$ in (2.1). Then $\mathbf{L} \equiv L_H + L_A$, $[L_H, L_A] = 0$. As $-L_H$ is positive, all the eigenvalues of $\mathbf{L}$ have negative real parts. They also have nonzero imaginary parts provided by the anti-Hermitian operator L_A. Thus, $q(t, \mathbf{x})$ relaxes oscillatorily to zero for any initial condition $q(t, \mathbf{0})$. Oscillations and decay to zero are independent of each other, which characterizes type II states.[8] The H-theorem for the linear FPE (2.1) with $\langle \mathbf{x}(t) \rangle \equiv \mathbf{0}$ also implies that (2.3) is stable.[9] Let us define

$$H(p, p_s) \equiv -\int p(t, \mathbf{x}) \ln[p(t, \mathbf{x})/p_s(\mathbf{x})]\, d\mathbf{x} \tag{2.5}$$

The relative entropy $H(p, p_s)$ is a Liapunov functional for (2.1) with $\langle \mathbf{x}(t) \rangle \equiv \mathbf{0}$ [$H(p, p_s) < 0$ if $p \neq p_s$, $H(p_s, p_s) = 0$, and $dH(p, p_s)/dt \geqslant 0$. Any initial $p(0, \mathbf{x})$ will therefore relax roward $p_s(\mathbf{x})$ if $\langle \mathbf{x}(t) \rangle \equiv \mathbf{0}$.

In general $\langle \mathbf{x}(t) \rangle \neq \mathbf{0}$ and (2.3) may be unstable. The linear stability analysis of (2.3) may be done as in the case of the model (1.1).[5] The result in dimensionless form is given by the following equation:

$$A = \tfrac{1}{2}\theta^2 \left\{ 1 - \theta^{-1} \exp(-A^2/\theta^2) \left[\int_{-A/\theta}^{\infty} \exp(-r^2)\, dr \right]^{-1} \right\} \tag{2.6a}$$

$$A \equiv \alpha/J - 1 \tag{2.6b}$$

$$\theta = (2F)^{1/2}/J \tag{2.6c}$$

The regions of instability of (2.3) are shaded in Fig. 1. Equation (2.6) is obtained in the Appendix. For A at the critical line (2.6), there is either a sub- or a supercritical Hopf bifurcation.[10] Below the critical line, (2.3) is stable. Above (2.6), the following one-oscillator probability density is asymptotically stable (see below):

$$p_0(t, \mathbf{x}) = Z_0^{-1} \exp[-\phi(\mathbf{x}) + 2JF^{-1}\mathbf{x} \cdot \langle \mathbf{x}(t) \rangle_0] \tag{2.7a}$$

$$d\langle \mathbf{x}(t) \rangle_0 / dt = -\langle \mathbf{x}(t)^{\dagger} \rangle_0 \tag{2.7b}$$

$$Z_0 = \int \exp[-\phi(\mathbf{x}) + 2JF^{-1}\mathbf{x} \cdot \langle \mathbf{x}(t) \rangle_0]\, d\mathbf{x} \tag{2.7c}$$

Equation (2.7b) leaves the amplitude and phase of the oscillations of the mean value $\langle \mathbf{x}(t) \rangle_0$ unspecified. The amplitude $\langle \mathbf{x}(t) \rangle_0^2$ is implicitly given by the equation

$$\delta \ln Z_0 / \delta \langle \mathbf{x}(t) \rangle_0 = 2JF^{-1} \langle \mathbf{x}(t) \rangle_0 \tag{2.7d}$$

Here and in what follows, the subscript zero indicates that the averages are with respect to the oscillatory state (2.7). That (2.7a)–(2.7c) is a solution of (2.1) follows from direct substitution in (2.1). By taking a functional derivative of $\ln Z_0$ with respect to $\langle \mathbf{x}(t) \rangle_0$, we find (2.7d). Exactly at the bifurcation line (2.6), the functional derivative of (2.7d) with respect to $\langle \mathbf{x}(t) \rangle_0$ is singular:

$$\det[\delta^2 \ln Z_0 / \delta \langle \mathbf{x}(t) \rangle_0 \, \delta \langle \mathbf{x}(t) \rangle_0 - (2J/F)\, \mathbf{1}] = 0 \tag{2.8}$$

At the bifurcation line $\langle \mathbf{x}(t) \rangle_0 = \mathbf{0}$, the determinant in (2.8) is equal to $[JF^{-1} \langle \mathbf{x}(t) \rangle_0 - 1]^2$, and (2.8) also yields (2.6), as shown in the Appendix. Near the bifurcation line (2.6), we can find an approximation to the bifurcating solution (2.7) by: (a) expanding Z_0 in powers of $\langle \mathbf{x}(t) \rangle_0$ and solving (2.7d) iteratively; and (b) by adapting standard Hopf bifurcation calculations[10] to the nonlinear FPE (2.1). We shall sketch both methods below, as both shed light on different aspects of the oscillator synchronization: method (a) emphasizes the "thermodynamics" of the nonequilibrium phase transition and its similarity with the Curie–Weiss

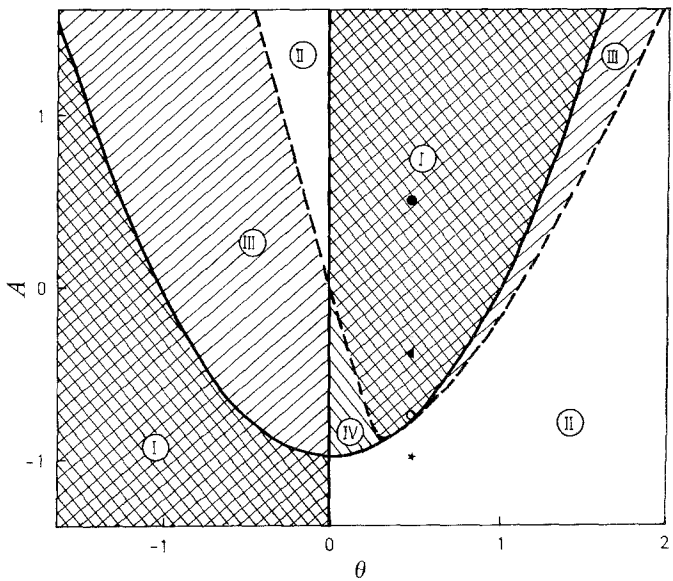

Fig. 1. Stability diagram for the stationary solution (2.3) as given exactly by Eq. (2.6) and approximately by the Gaussian truncation (GT). Region I: $p_s(\mathbf{x})$ is unstable according to both (2.6) and GT. Region II: $p_s(\mathbf{x})$ is stable according to both (2.6) and GT. Region III: $p_s(\mathbf{x})$ is unstable, but GT yields the opposite result. Region IV: $p_s(\mathbf{x})$ is stable, but GT yields the opposite result.

model[11] and the theory of Landau.[12] By using method (b), we build the bifurcating solution and find its stability at the same time. The result is

$$p_0(t, \mathbf{x}) = Z_0^{-1} \exp\{-\phi(\mathbf{x}) + [2JF^{-1}(A_c + 1 - \tfrac{1}{2}\theta^2)/(\theta^2 - A_c - 1)]^{1/2} \times (A - A_c)^{1/2}(y \cos t - x \sin t) + O(|A - A_c|)\} \tag{2.9}$$

Here we have taken A (or α in dimensional form) as our bifurcation parameter. A_c is the value of A at the critical line (2.6). Let θ^* be the point where the curve (2.6) intersects the parabola $A + 1 = \theta^2$, as shown in Fig. 1. For $\theta > \theta^*$ the oscillatory density (2.9) exists only if $A > A_c$, and it is asymptotically stable except for constant phase shifts, as the Hopf theorem shows.[10] For $\theta < \theta^*$ the oscillatory density (2.9) exists only if $A < A_c$, and it is then unstable. We have determined the bifurcation diagram near $\theta = \theta^*$ and near the minimum of the instability region in Fig. 1. Figure 2 shows different sections of the amplitude of the oscillations versus A and θ.

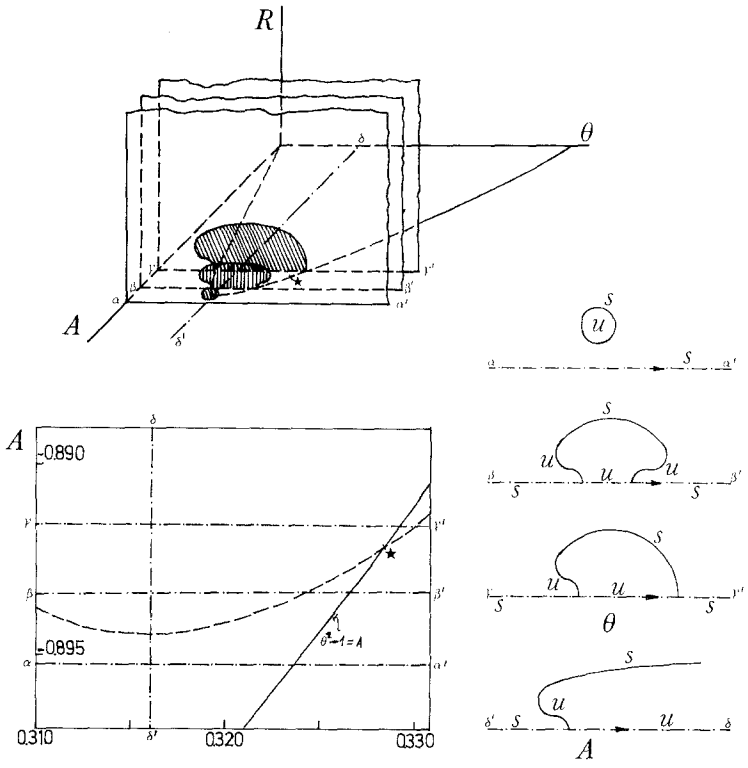

Fig. 2. Bifurcation diagrams for $R \equiv [\langle \mathbf{x}(t) \rangle_0^2]^{1/2}$ versus A and θ.

The point on the bifurcation line with $\theta = \theta^*$ separates a region where there is a second-order phase transition (for $A \geqslant A_c$) to the oscillatory state from a region where the phase transition is first order.

2.1. Thermodynamic Method. Role of the Entropy

Let us derive (2.9) by method (a). It is not hard to see that Z_0 depends only on the amplitude of the mean value (2.7b), and thus it is time-independent. We just write the integral in (2.7c) in polar coordinates and use the integral representation of the modified Bessel function $I_0(x)$.[13] The result is

$$Z_0 = Z\langle I_0(2JF^{-1}[\langle \mathbf{x}(t)\rangle_0^2 \mathbf{x}^2]^{1/2})\rangle_s \tag{2.10}$$

Here the subscript "s" means that we average with respect to the stationary probability density (2.3). Physically, we may interpret $-F \ln Z_0$ as the nonequilibrium free energy of the state (2.7). By expanding the Bessel function in powers of its argument, we obtain a polynomial approximation of the free energy similar to Landau's free energy.[12] As the phase of the oscillations has disappeared from (2.10), we can simplify our formulas by reparametrizing the free energy, thereby eliminating one parameter:

$$\mathbf{y} = \mathbf{x}(J/F)^{1/2}, \qquad \langle \mathbf{y}(t)\rangle_0 = (J/F)^{1/2}\langle \mathbf{x}(t)\rangle_0 \tag{2.11}$$

With this dimensionless variable, the partition function in the stationary state (2.3) becomes

$$\begin{aligned} Z &= \int \exp\{(JA - \tfrac{1}{2}\mathbf{x}^2)\mathbf{x}^2/F\}\, d\mathbf{x} \\ &= (F/J) \int \exp[(A - \tfrac{1}{4}\theta^2\mathbf{y}^2)\mathbf{y}^2]\, d\mathbf{y} \\ &\equiv (F/J)\,\zeta \end{aligned} \tag{2.12}$$

We now define the dimensionless free energy as $-\ln \zeta_0$, where

$$\begin{aligned} \zeta_0 &\equiv \int \exp[(A - \tfrac{1}{4}\theta^2\mathbf{y}^2)\,\mathbf{y}^2]\, I_0(2[\langle \mathbf{y}(t)\rangle_0^2\, \mathbf{y}^2]^{1/2}\, d\mathbf{y} \\ &\equiv \zeta\langle I_0(2[\langle \mathbf{y}(t)\rangle_0^2\, \mathbf{y}^2]^{1/2})\rangle_s \end{aligned} \tag{2.13}$$

Notice that we have one less parameter than in (2.10). Direct reparametrization of (2.1) does not change the number of independent parameters, it merely alters their placement in the equation. In fact, (2.1) is

dimensionless: we have fixed the time scale by imposing the frequency of the oscillations to be unity. The self-consistency condition (2.7d) now becomes

$$\langle \mathbf{y}(t)\rangle_0 = \delta \ln \zeta_0/\delta\langle \mathbf{y}(t)\rangle_0$$

i.e.,

$$\begin{aligned}\langle \mathbf{y}(t)\rangle_0 = \langle \mathbf{y}(t)\rangle_0\{&\langle \mathbf{y}^2\rangle_s + \tfrac{1}{2}[\langle (\mathbf{y}^2)^2\rangle_s - 2\langle \mathbf{y}^2\rangle_s^2]\langle \mathbf{y}(t)\rangle_0^2 \\ &+ [\langle (\mathbf{y}^2)^3\rangle_s + 12\langle \mathbf{y}^2\rangle_s^3 \\ &- 9\langle \mathbf{y}^2\rangle_s\langle (\mathbf{y}^2)^2\rangle_s][\langle \mathbf{y}(t)\rangle_0^2]^2/12 \\ &+ O([\langle \mathbf{y}(t)\rangle_0^2]^3)\}\end{aligned} \tag{2.14}$$

Equation (2.14) is obtained by expanding the free energy (2.13) in powers of the amplitude of the oscillations, $\langle \mathbf{y}(t)\rangle_0^2$. Equation (2.14) is different from Landau's theory, where the equilibrium value of the order parameter minimizes the free energy.[12] As a consequence, we will see that the entropy is a bad measure of the "disorder" in our problem. It turns out that the entropy in the ordered phase, where $\langle \mathbf{y}(t)\rangle_0 \neq 0$, can be either smaller or larger than the entropy in the disordered phase, (2.3), where $\langle \mathbf{y}\rangle_s = 0$.

To find the amplitude of the oscillations, we now expand the coefficients of the series in (2.14) in powers of $(A - A_c)$, where A_c is the value of A at the bifurcation line (2.6). The result is

$$\begin{aligned}\langle \mathbf{y}(t)\rangle_0 = \langle \mathbf{y}(t)\rangle_0\{&1 + [2\theta^{-2}(A_c + 1) - 1](A - A_c) \\ &+ [\theta^{-2}(1 + A_c) - 1](\langle \mathbf{y}(t)\rangle_0^2\} + \cdots\end{aligned} \tag{2.15}$$

We have kept the leading order terms in $(A - A_c)$ and ignored higher order terms in the mean value of $\mathbf{y}$, which is legitimate when $1 + A_c - \theta^2 \neq 0$. The nonzero solution of (2.15) yields (2.9). Similarly, we can use θ as our bifurcation parameter. We find

$$\begin{aligned}\langle \mathbf{y}(t)\rangle_0 = \langle \mathbf{y}(t)\rangle_0\{&1 + [A_c - 1 - 2A_c(A_c + 1)/\theta_c^2](\theta - \theta_c)/\theta_c \\ &+ [\theta_c^{-2}(1 + A_c) - 1]\langle \mathbf{y}(t)\rangle_0^2\} + \cdots\end{aligned} \tag{2.16}$$

When $\theta = \theta^*$, $1 + A_c - \theta^2$ is zero, and we have to retain higher order terms in (2.14). At the minimum of the curve (2.6), $\theta_c = \theta_m$, the coefficient of $(\theta - \theta_c)$ in (2.16) is zero, and we also run into trouble. As θ^* and θ_m are quite close to each other, we expect that an analysis of (2.14) in the neighborhood of θ_m will tell us what happens in the neighborhood of θ^* as

well. Let us retain all the terms displayed in (2.14) and expand their coefficients in powers of $(A - A_m)$ and of $(\theta - \theta_m)$. By imposing that

$$R^4 \equiv (\langle \mathbf{y}(t) \rangle_0^2)^2 \sim (A - A_m) \sim (\theta - \theta_m)^2$$

in the resulting equation, we find the following dominant balance:

$$R^4 + 3\{[-\tfrac{1}{2}(1 + A_m) + (1 - A_m)(\theta - \theta_m)/\theta_m] R^2 - (A - A_m) + \tfrac{1}{2}(1 - A_m)^2(\theta - \theta_m)^2/(1 + A_m)\}(2 + A_m)^{-1} = 0 \tag{2.17}$$

We have kept the term $(1 - A_m)(\theta - \theta_m) R^2/\theta_m$ in (2.17) because of the smallness of $(1 + A_m)$. By equating the coefficient of R^2 to zero, we get an approximate θ^* that differs from the exact one by terms of order $(1 + A_m)^2$. The solutions of (2.17) for different values of A and θ allow us to build Fig. 2, where "s" means stable and "u" means unstable, with the obvious meanings for both the stationary and oscillatory solutions. The stability assignements have been made according to the principle of exchange of stability in the Hopf bifurcation.[10] See below for a different criterion. A complete classification and unfoldings of degenerate Hopf bifurcations have been established (for codimension $\leqslant 3$) by Golubitsky and Langford.[14]

Let us now calculate the entropy difference between the oscillatory and the stationary states for equal values of the parameters. By using (2.3), (2.7), and (2.10), together with the dimensionless variables and free energies (2.11)–(2.13), we obtain:

$$\Delta S \equiv S_0 - S_s \equiv -\tfrac{1}{2}[\langle \mathbf{y}^2 \rangle_0 - \langle \mathbf{y}^2 \rangle_s] - \langle \mathbf{y} \rangle_0^2 + \ln \langle I_0(2(\langle \mathbf{y} \rangle_0^2 \mathbf{y}^2)^{1/2}) \rangle_s \tag{2.18}$$

By expanding the terms in (2.18) in powers in $\langle \mathbf{y} \rangle_0^2$, we get the following approximate expression:

$$\Delta S = \langle \mathbf{y} \rangle_0^2 [-1 - A/\theta^2 + (1 - A^2\theta^{-2} + \tfrac{1}{2}A\langle \mathbf{y}^2 \rangle_s] + O[(\langle \mathbf{y} \rangle_0^2)^2] \tag{2.19}$$

Near the bifurcation line (2.6) we can use (2.9) and expand (2.19) in powers of $(A - A_c)$. The result is

$$\Delta S = -\tfrac{1}{2}(2 + 2A_c - \theta^2)^2 A_c (A - A_c) \theta^{-2} (\theta^2 - 1 - A_c)^{-1} + O(|A - A_c|^{3/2}) \tag{2.20}$$

For $\theta > \theta^*$ and (A, θ) *near* (2.6) the entropy of the (stable) oscillatory state is larger than that of the (unstable) stationary state if $A < 0$. The opposite holds true if $A > 0$: the entropy is larger at the stationary state. We see that the entropy is neither a measure of the disorder nor a sign of stability. A similar calculation shows that the free energy is not a ther-

modynamic potential, as $\ln(Z/Z_0)$ can be positive or negative. For $\theta < \theta^*$ the stable oscillatory state has a nonzero order parameter $\langle \mathbf{y} \rangle_0^2$ even at the bifurcation line (2.6). We have a first-order phase transition with a positive "latent heat" given by F times ΔS,

$$\begin{aligned} \Delta S \sim & -(1 + A_c - \tfrac{1}{2}\theta_c^2)(A_c/\theta_c^2) R_c^2 \\ = & -3A_c(1 + A_c - \tfrac{1}{2}\theta_c^2)[\theta_c^2(2 + A_m)]^{-1}[\tfrac{1}{2}(1 + A_m) \\ & -(1 - A_m)(\theta_c/\theta_m - 1)] \end{aligned} \tag{2.21}$$

(2.21) is obtained by noticing that the independent term in (2.17) vanishes at the bifurcation line (2.6). At the bifurcation line we can thus compute the amplitude from (2.17) and substitute in (2.19), thereby getting (2.21).

2.2. Evolution toward the Oscillatory State

Let us use A as our bifurcation parameter. From Fig. 1 we see that for $A < A_c$ the stationary density is stable, while it is unstable if $A > A_c$ (for $\theta \geqslant 0$). We shall build the bifurcating time-periodic density by means of the two-time method of Kogelman and Keller.[15] Use of this small-amplitude method in (2.1) yields a vertical bifurcation to all orders in the expansion. In physical terms, nonlinear response theory fails to all orders. Thus, we apply the method to Eq. (2.2) for $\Phi(t, \mathbf{x}) = -\ln p(t, \mathbf{x})$. Let us define a small parameter ε to measure the distance to the bifurcation point A_c:

$$A = A_c + \varepsilon A_1 + \varepsilon^2 A_2 + O(\varepsilon^3) \tag{2.22}$$

For A near its critical value, we try an expansion of the following form for $\Phi(t, \mathbf{x})$:

$$\Phi(t, \mathbf{x}) = \ln Z + \phi(\mathbf{x}) + \varepsilon \Psi_1(s, \tau, \mathbf{x}) + \varepsilon^2 \Psi_2(s, \tau, \mathbf{x}) + \varepsilon^3 \Psi_3(s, \tau, \mathbf{x}) + O(\varepsilon^4) \tag{2.23}$$

Here the order-zero term is (2.3), and we have introduced two independent time scales $s = t$ and $\tau = (A - A_c)t$. These separated scales measure, respectively, the oscillations of frequency 1 and the slow evolution of the amplitude of these oscillations toward their final value. We now insert (2.22) and (2.23) in (2.2), together with $\partial_t = \partial_s + (A - A_c)\partial_\tau$. A hierarchy of linear equations results:

$$\mathbf{M}\Psi_1 \equiv [p_s(\mathbf{x})]^{-1/2}(\partial_s - \mathbf{L}_c)\{[p_s(\mathbf{x})]^{1/2}\Psi_1\} = 0 \tag{2.24a}$$

$$\begin{aligned} \mathbf{M}\Psi_2 = & A_1(\cdots)\Psi_1 - \tfrac{1}{2}F(\nabla\Psi_1)^2 - \tfrac{1}{2}J\nabla\phi \cdot \langle \mathbf{x}, p_s(\mathbf{x})\Psi_1^2 \rangle \\ & + J\nabla\Psi_1 \cdot \langle \mathbf{x}, p_s(\mathbf{x})\Psi_1 \rangle \end{aligned} \tag{2.24b}$$

$$\begin{aligned}\mathbf{M}\Psi_3 = & -A_2[\partial_\tau - J\mathbf{x}\cdot\nabla + 2J^2F^{-1}\mathbf{x}\cdot\langle \mathbf{x}p_s(\mathbf{x}),\cdot\rangle \\ & - J^2F^{-1}\nabla\phi\cdot\langle \mathbf{x}(\mathbf{x}^2 - F/J)\, p_s(\mathbf{x}),\cdot\rangle]\,\Psi_1 + A_1(\cdots) \\ & - F\nabla\Psi_1\cdot\nabla\Psi_2 - J\nabla\phi\cdot\langle \mathbf{x},\, p_s(\mathbf{x})\,\Psi_1\Psi_2\rangle \\ & + J\nabla\Psi_2\cdot\langle \mathbf{x},\, p_s(\mathbf{x})\Psi_1\rangle + J\nabla\Psi_1\cdot\langle \mathbf{x},\, p_s(\mathbf{x})\Psi_2\rangle \\ & + J\nabla\phi\cdot\langle \mathbf{x},\, p_s(\mathbf{x})\Psi_1^3\rangle/6 - \tfrac{1}{2}J\nabla\Psi_1\cdot\langle \mathbf{x},\, \Psi_1^2\rangle \end{aligned} \tag{2.24c}$$

In addition, we have the following normalization conditions:

$$\langle p_s(\mathbf{x}),\, \Psi_1\rangle = 0 \tag{2.25a}$$

$$\langle p_s(\mathbf{x}),\, \Psi_2\rangle = \tfrac{1}{2}\langle p_s(\mathbf{x}),\, \Psi_1^2\rangle \tag{2.25b}$$

$$\langle p_s(\mathbf{x}),\, \Psi_3 - \Psi_1\Psi_2 + \Psi_1^3/6\rangle = 0 \tag{2.25c}$$

The solution of (2.24a) and (2.25a) is

$$\Psi_1(s,\tau,\mathbf{x}) = (ix + y)\, e^{is}a(\tau) + \text{c.c.} \tag{2.26}$$

We need to find an evolution equation for $a(\tau)$, so as to determine the leading order approximation to the bifurcating solution. This we do by solving successively the remaining equations in (2.24) and (2.25). For the inhomogeneous equation (2.24b) to be solvable, its right side must be orthogonal to the solution of the adjoint equation to (2.24a). This Fredholm alternative condition yields $A_1 = 0$. Anticipating this result, we did not explicitly write the terms multiplying A_1 in (2.24). The solution of (2.24b) and (2.25b) is

$$\Psi_2(s,\tau,\mathbf{x}) = FJ^{-1}|a(\tau)|^2$$

Insertion of this and of (2.26) in (2.24c) plus the Fredholm alternative yield the following equation for $a(\tau)$:

$$da(\tau)/d\tau = \nu J[(A_c + 1 - \tfrac{1}{2}\theta^2)\, JA_2 + \tfrac{1}{2}F(A_c + 1 - \theta^2)|a(\tau)|^2]\, a(\tau) \tag{2.27}$$

$$\nu = \tfrac{1}{2}\langle \psi,\, [p_s(\mathbf{x})]^{1/2}(i,\, 1)\cdot\nabla\Psi_1\rangle \tag{2.28}$$

The function ψ in (2.28) solves the adjoint equation $\mathbf{L}^\dagger\psi = i\psi$. The trouble with (2.27) is that we have not been able to solve the adjoint equation, and thus we ignore the explicit form of ν. The term ν is proportional to the derivative with respect to A (at A_c) of the eigenvalue that crosses the

imaginary axis at the critical line. From Fig. 1 we infer $v > 0$ for the part of the critical line with $\theta \geqslant 0$. For the region of Fig. 1 where $\theta \geqslant 0$ and the stationary solution is unstable, the sign of the coefficient of $a(\tau)$ in the right side of (2.27) is positive, and it is negative just below the critical line. The sign of the coefficient of the cubic term is always negative near the critical line. Then, any initial condition of (2.27) tends toward $a = 0$ for $A_2 < 0$ $(A < A_c)$, and it tends toward

$$a(\infty) = [2JF^{-1}A_2(A_c + 1 - \tfrac{1}{2}\theta^2)/(\theta^2 - A_c - 1)]^{1/2}$$

otherwise. This ends the derivation of (2.9), and proves the claims about the asymptotic stability of the bifurcating time-periodic density. We will not consider here how to extend the two-time method to the degenerate case (2.17). It is not hard to do it knowing the result beforehand: instead of zero, we should get a right side proportional to $R^{-1}\, dR/d\tau$ in (2.17). From this time-dependent equation we can show the validity of the principle of exchange of stability used to draw Fig. 2.

As indicated by (2.10), the "thermodynamics" of our state (2.7) (given by $\ln Z_0$) depends only on $\langle \mathbf{x}(t) \rangle_0^2$, and therefore is independent of the time. The "kinetic theory," however, might not register a monotone trend of the entropy toward its value in the state (2.7). The time derivative of the relative entropy $H(p, p_0)$ defined by (2.5) is

$$\begin{aligned} dH(p, p_0)/dt = {} & \tfrac{1}{2}F \int p(t, \mathbf{x}) \{\nabla \ln[p(t, \mathbf{x})/p_0(t, \mathbf{x})]\}^2 \, d\mathbf{x} \\ & - J(\langle \mathbf{x}(t) \rangle - \langle \mathbf{x}(t) \rangle_0) \\ & \times \int p(t, \mathbf{x}) \nabla \{\ln[p(t, \mathbf{x})/p_0(t, \mathbf{x})]\} \, d\mathbf{x} \end{aligned} \tag{2.29}$$

Because of the second term in the right side of (2.29), $dH(p, p_0)/dt$ might change signs during the rise of the relative entropy to zero, and thus $H(p, p_0)$ would not be a Liapunov function. As we saw before, the absolute entropy is not a good indicator of stability (thermodynamic potential) either for our nonlinear FPE. The question of what is the Liapunov function for our equation remains unanswered.

It is not clear whether the features of the model (1.3) so far described are general for the class of mean-field models considered here and in Ref. 7. In fact, we can perform so many explicit calculations only because both the rotation and the mean-field coupling terms are linear. Relaxation of any of these assumptions complicates even the linear stability analysis (2.4).

3. NUMERICAL RESULTS

We describe here the relaxation toward the stationary or time-periodic probability densities from a given initial condition. It is not possible to do so analytically. We thus resort to numerical methods. Following previous work,[3,6] we use a cumulant expansion to describe the relaxation to the stable state. The cumulant-generating function $W(t, \xi)$ is defined by

$$\exp[W(t, \xi)] = \int \exp[\xi \cdot \mathbf{x}]\, p(t, \mathbf{x})\, d\mathbf{x}, \qquad \xi \equiv (\xi, \eta) \tag{3.1}$$

The (one-time) cumulants are the coefficients of the expansion of $W(t, \xi)$ in powers of ξ. From (3.1) and the nonlinear FPE (2.1), we find the following evolution equation for $W(t, \xi)$:

$$\begin{aligned} \partial_t W(t, \xi) = {} & \tfrac{1}{2} F \xi^2 + J\xi \cdot \langle \mathbf{x}(t) \rangle + \xi \cdot ((\alpha - J) \nabla W(t, \xi) \\ & - \nabla \Delta W(t, \xi) - \nabla W(t, \xi) \{ \Delta W(t, \xi) + [\nabla W(t, \xi)]^2 \} \\ & - \nabla W(t, \xi) \cdot \nabla \nabla W(t, \xi)) + \xi^{\dagger} \cdot \nabla W(t, \xi) \end{aligned} \tag{3.2}$$

Let us expand $W(t, \xi)$ in powers of ξ and equate both sides of (3.2) term by term. The result is a hierarchy of coupled evolution equations for the cumulants. In the equations for each cumulant we find cumulants of higher order. Thus, to solve the hierarchy is not an easier task than solving (2.1) or (3.1). In practice, we will close this hierarchy by demanding that all cumulants of order higher than two be zero. This closure assumption is the Gaussian truncation (GT), a "dishonest" perturbation method,[16] which does not correspond to any expansion in a small parameter. In spite of its character, the GT works pretty well in a wide region of the parameter space, as we report below. More complete discussions of this and related approximations can be found in Refs. 3, 5, and 6.

The evolution equations for the first-order $[c_1 = \langle x(t) \rangle, c_2 = \langle y(t) \rangle]$ and second-order $[c_{11} = \langle x(t)^2 \rangle - \langle x(t) \rangle^2, \quad c_{22} = \langle y(t)^2 \rangle - \langle y(t) \rangle^2, c_{12} = \langle x(t)\, y(t) \rangle - \langle x(t) \rangle \langle y(t) \rangle]$ cumulants are

$$dc_1/dt = c_1[\alpha - (c_1^2 + c_2^2) - 3c_{11} - c_{22}] - c_2(1 + 2c_{12}) \tag{3.3a}$$

$$dc_2/dt = c_2[\alpha - (c_1^2 + c_2^2) - 3c_{22} - c_{11}] + c_1(1 - 2c_{12}) \tag{3.3b}$$

$$\begin{aligned} \tfrac{1}{2}\, dc_{11}/dt = {} & c_{11}(\alpha - J - 3c_1^2 - c_2^2 - c_{22}) \\ & - 3c_{11}^2 - c_{12}(1 + 2c_1 c_2 + 2c_{12}) + \tfrac{1}{2} F \end{aligned} \tag{3.3c}$$

$$\tfrac{1}{2}\, dc_{22}/dt = c_{22}(\alpha - J - 3c_2^2 - c_1^2 - c_{11}) - 3c_{22}^2 + c_{12}(1 - 2c_1 c_2 - 2c_{12}) + \tfrac{1}{2}F \tag{3.3d}$$

$$\tfrac{1}{2}\, dc_{12}/dt = c_{12}[\alpha - J - 3(c_{11} + c_{22}) - 2(c_1^2 + c_2^2)] + \tfrac{1}{2}(c_{11} - c_{22}) - c_1 c_2 (c_{11} + c_{22}) \tag{3.3e}$$

The bifurcation line may be computed from (3.3). The result is $\theta^2 = A + 1$. In Fig. 1 we have compared this curve with the exact bifurcation line, Eq. (2.6). This comparison gives an indication of the region where the GT (3.3) approximates the true cumulants: We expect the GT to work far from the A axis, in regions I and II of Fig. 1, where both the exact formula (2.6) and the truncation assign stability to the same solution (stationary or oscillatory).

We have solved (3.3) by a fourth-order Runge-Kutta algorithm together with predictor-corrector formulas. The results are depicted in Figs. 3 and 4. After a transient regime (whose duration increases when we approach the critical line: the critical slowing down), the system settles down to the stable solution, oscillatory (Figs. 3a and 3b) or stationary (Fig. 3c).

The frequency of the oscillations of the time-periodic solution is one for the first cumulants and two for the second ones. It is independent of the parameters A and θ, as indicated by the exact solution (2.7). The amplitude of the oscillations depends on the values of A and θ. This can be seen in Fig. 4, where we plot them against A for fixed θ. Notice that Fig. 4 is the typical diagram for a second-order phase transition.

We have also analyzed the information entropy $S = -\int p(t, \mathbf{x}) \ln p(t, \mathbf{x})\, d\mathbf{x}$ in the Gaussian approximation. The result is

$$S_G = 1 + \ln(2\pi |c|^{1/2}), \qquad \text{where} \quad c = c_{11} c_{22} - c_{12}^2 \tag{3.4}$$

With the numerical solution of (3.3), we can easily evaluate this entropy. The results for different situations are plotted in Fig. 5. We have displayed the evolution of S_G for the parameter values corresponding to three of the points marked in Fig. 1: The dot and the triangle are in region I, where the oscillatory solution is stable and the stationary one is unstable. The star in Fig. 1 is in region II, where the stationary solution is stable and there is no oscillatory solution. Similar results are obtained for different initial conditions, including the case of a delta-function initial probability ($c = 0$).

S_G presents a nonmonotone behavior in its evolution toward its value in the corresponding stable state. This result is surely spurious, an artifact

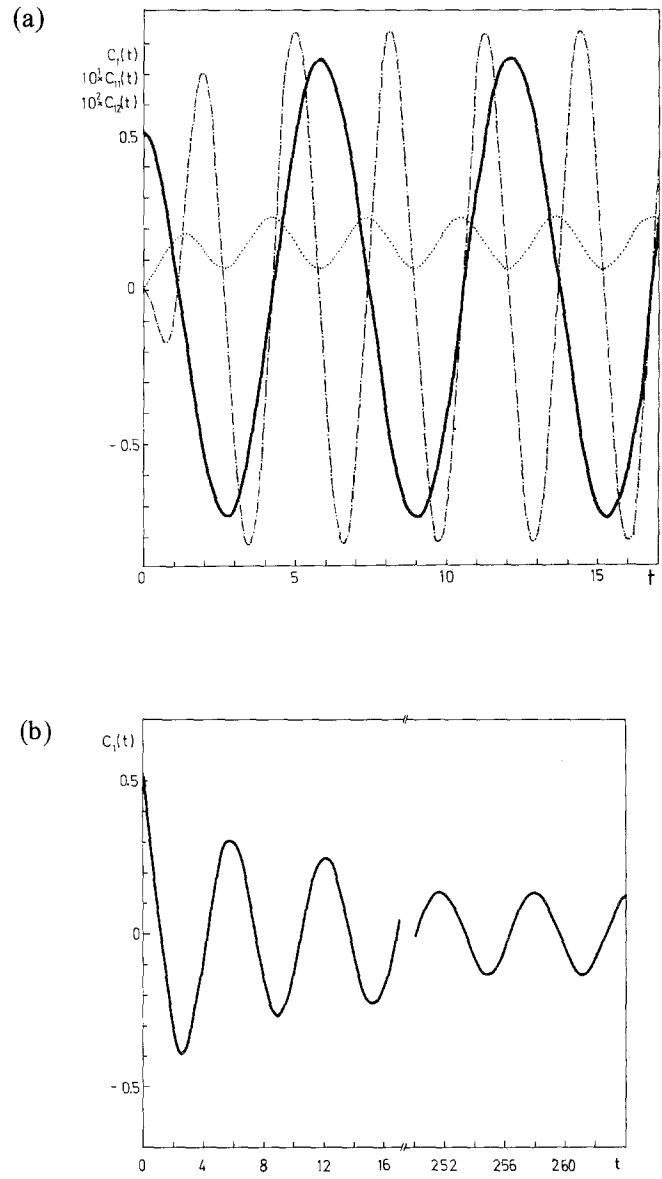

Fig. 3. (a) Time evolution of (—) $c_1(t)$, (– –) $c_{12}(t)$, (···) $c_{11}(t)$, in the Gaussian approximation. The aparameter values $\theta = A = 0.5$, $F = 0.02$ (dot in Fig. 1) correspond to a stable oscillatory state. The initial conditions are $c_1(0) = 0.51$, $c_2(0) = 0.24$, $c_{11}(0) = c_{12}(0) = c_{22}(0) = 0$. (b) Time evolution of $c_1(t)$ for $\theta = 0.5$, $A = -0.72$ (open circle in Fig. 1), $F = 0.02$, and the same initial conditions as in part (a). Notice the break in the time axis.

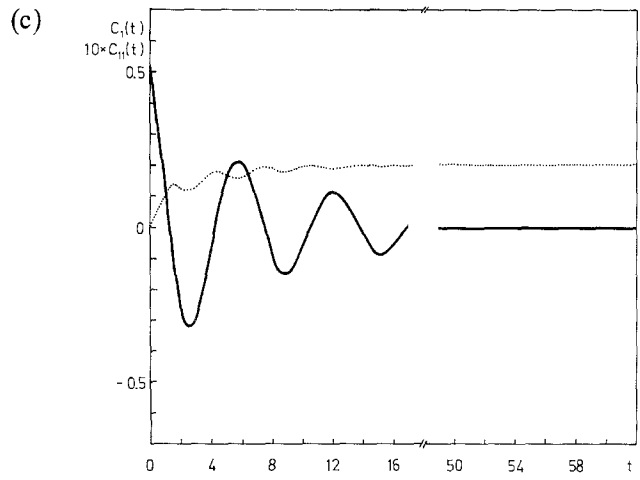

Fig. 3 (*continued*) (c) Time evolution of $c_1(t)$ and of $c_{11}(t)$ for $\theta = 0.5$, $A = -1$, $F = 0.02$ (star in Fig. 1), and the same initial conditions as in part (a). Notice the break in the time axis.

due to the Gaussian truncation. In fact, the nonmonotone behavior is more noticeable the farther the initial condition is from the stable state. But for such initial conditions non-Gaussian behavior is expected.[6] When we choose initial conditions closer to the oscillatory state, the entropy tends monotonically to its final value. This is also the case as we approach the

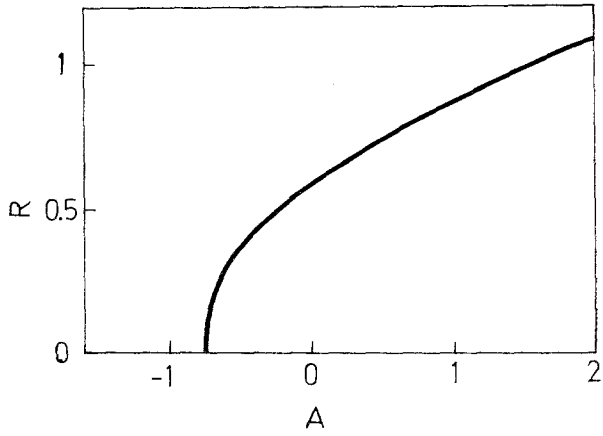

Fig. 4. Plot of the amplitude of the oscillations $R = [c_1(t)^2 + c_2(t)^2]^{1/2}$ versus A in the Gaussian approximation for fixed $\theta = 0.5$, $F = 0.02$.

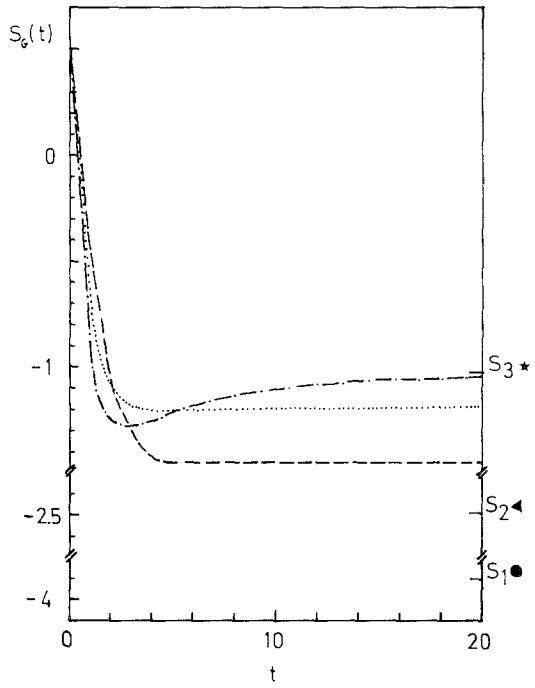

Fig. 5. Plot of the time evolution of S_G for $F=0.02$, $\theta=0.5$, and three different values of A in the oscillatory state: (– –) $A=0.5$ (dot in Fig. 1); (···) $A=-0.4$ (triangle in Fig. 1); (– –) $A=-1$ (star in Fig. 1). For each value of A, we have calculated the information entropy with the exact $p_s(\mathbf{x})$ of Eq. (2.3). The results are the lines S_1, S_2, S_3, which correspond to $A=0.5$, -0.4, and -1, respectively. The initial conditions are as follows: $c_1(0)=0.51$, $c_2(0)=0.24$, $c_{11}(0)=c_{22}(0)=0.1$, $c_{12}(0)=0.001$. Notice the break in the entropy axis.

critical line from above. We have chosen to approach the critical line in the region where the Gaussian line of Fig. 1 is tangent to the exact (dashed) line in Fig. 1. In this way we make sure that the Gaussian truncation does not falsify the results. In Fig. 5 we have also plotted the entropy of the stationary state. We observe that it is smaller than the entropy of the stable oscillatory state for θ and A corresponding to the points marked by the dot and by triangle in Fig. 1. This is not the case for the values corresponding to the circle. This can be understood by means of Eq. (2.20), as $\theta^2 < A_c + 1$, $A > A_c$, and $A_c < 0$. For values of θ and A corresponding to the star in Fig. 1, the entropy tends toward its value at the stable stationary state.

4. SUMMARY

We have analyzed a simple model of self-synchronization of large populations of nonlinear oscillators. Each oscillator is in contact with a thermal bath at temperature F and interacts linearly with the mean field of all the others. The oscillators feel no external influence besides the bath's. In the thermodynamic limit we have described the nonequilibrium phase transitions from a stationary to a time-periodic probability density. We have found first- and second-order phase transitions according to the values of the parameters. Furthermore, the information entropy is constant in each stable state (stationary or time-periodic probability density, according to the values of the parameters). We can define, even in the oscillatory state, constant free and internal energies following the usual rules of equilibrium statistical mechanics. However, they are not thermodynamic potentials in the usual sense, as they do not indicate which state is stable. Furthermore, fully to describe the system, one really needs the one-oscillator probability density: The Einstein relation between entropy and probability density does not hold in the oscillatory state.

APPENDIX. STABILITY ANALYSIS OF $p_s(\mathbf{x})$

Our analysis follows that by Dawson.[5] To the equation (2.4a) for a disturbance about the stationary state $p_s(\mathbf{x})$ we associate the eigenvalue problem

$$\lambda w(\mathbf{x}) = \mathbf{L} w(\mathbf{x}) \tag{A.1}$$

Equation (A.1) is obtained by inserting $q(t, \mathbf{x}) = e^{\lambda t} w(\mathbf{x})$ in the linearization of (2.4a) about $q = 0$. Consider the linear operator $M = -L_H - L_A$. As the operators L_H and L_A commute, they have the same eigenfunctions. The eigenvalues of L_H and L_A are, respectively, the real and imaginary parts of the eigenvalues of M. The term $[p_s(\mathbf{x})]^{1/2}$ is the ground state of M and its corresponding eigenvalue is zero. All other eigenvalues have positive real parts. Let us call $\lambda_j(M)$ the jth eigenvalue of the operator M. The eigenvalues of positive operators are ordered by their magnitudes, starting from the corresponding to the ground state. We want to compare $\lambda_j(M)$ and $\operatorname{Re} \lambda_j(\mathbf{L})$. Let $\mathscr{H}_1$ be the three-dimensional Hilbert subspace of $L^2(\mathbb{R}^2)$ which is spanned by $[p_s(\mathbf{x})]^{1/2}$ and by $\mathbf{n} \cdot \mathbf{x}[p_s(\mathbf{x})]^{1/2}$ for any $\mathbf{n} \in \mathbb{R}^2$. Let $\mathscr{H}_1^{\perp}$ be the orthogonal subspace to $\mathscr{H}_1$. Let Π be the projector onto $\mathscr{H}_1^{\perp}$. Then,

$$\langle \Sigma, \Pi \mathbf{L} \Pi \Omega \rangle = -\langle \Sigma, M \Omega \rangle, \qquad \Sigma, \Omega \in \mathscr{H}_1^{\perp} \tag{A.2}$$

The scalar product $\langle , \rangle$ was defined in (2.4e). Equation (A.2) implies that

$$\begin{aligned} 0 < \lambda_1(-L_H) &= \inf\{-\langle \mu, L_H \mu \rangle : \|\mu\| = 1, \mu \in L^2(\mathbb{R}^2), \langle [p_s(\mathbf{x})]^{1/2}, \mu \rangle = 0\} \\ &\leqslant \inf\{-\langle \mu, L_H \mu \rangle : \|\mu\| = 1, \ \mu \in L^2(\mathbb{R}^2), \langle [p_s(\mathbf{x})]^{1/2}, \mu \rangle \\ &= \langle \mathbf{n} \cdot \mathbf{x}[p_s(\mathbf{x})]^{1/2}, \mu \rangle = 0\} \\ &= -\mathrm{Re}\, \lambda_1(\Pi \mathbf{L} \Pi) \end{aligned}$$

Thus, all the real parts of the eigenvalues of $\Pi \mathbf{L} \Pi$ are negative. There is then only one possibility of getting $[p_s(\mathbf{x})]^{1/2}$ to be unstable: finding eigenvalues of $\mathbf{L}$ that are not orthogonal to $\mathbf{n} \cdot \mathbf{x}[p_s(\mathbf{x})]^{1/2}$ and correspond to eigenvalues of positive real parts. Let us try $w(\mathbf{x}, \mathbf{n}) \equiv \mathbf{n} \cdot \mathbf{x}[p_s(\mathbf{x})]^{1/2}$ itself in Eq. (A.1). We get

$$\lambda \mathbf{n} \cdot \mathbf{x} = \tfrac{1}{2}(J\langle \mathbf{x}^2, p_s(\mathbf{x}) \rangle - F)\mathbf{n} \cdot \nabla \phi - \mathbf{x}^{\dagger} \cdot \mathbf{n} \tag{A.3}$$

Let us consider now the function $w(\mathbf{x}, (0, 1)) + iw(\mathbf{x}, (1, 0)) \equiv \Omega(\mathbf{x})$. By repeating the procedure that led to (A.3), we obtain

$$(\lambda - 1)\, \Omega(\mathbf{x}) = -(J\langle \mathbf{x}^2, p_s(\mathbf{x}) \rangle - F)(i\partial_x + \partial_y)\, [p_s(\mathbf{x})]^{1/2} \tag{A.4}$$

$\Omega(\mathbf{x})$ is an eigenfunction of $\mathbf{L}$ with eigenvalue i for those values of the parameters α, J, and F that make the right side of (A.4) equal to zero. Then $\Omega^*(\mathbf{x})$ corresponds to the eigenvalue $-i$. There is a Hopf bifurcation at those values of the parameters that make zero the right side of (A.4). The critical line at which the probability density bifurcates is given by

$$\int \mathbf{x}^2 p_s(\mathbf{x})\, d\mathbf{x} = F/J \tag{A.5}$$

After some algebra, we can derive (2.6) from this equation.

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